A NEW TEST OF COMPOUND SYMMETRY

by

S. N. ROY

INSTITUTE OF STATISTICS, UNIVERSITY OF NORTH CAROLINA

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1. **Summary**. If x_1 and x_2 have a bivariate normal distribution with a correlation coefficient ρ and the same standard deviations σ for both, then it is well known and easy to check that $x_1 + x_2$ and $x_1 - x_2$ are uncorrelated, which forms the basis of Pitman's well-known test of H_0: $\sigma_1 = \sigma_2$, for a bivariate normal population, in terms of the correlation coefficient r between $x_1 + x_2$ and $x_1 - x_2$ in a random sample of size, say, n, from this population. Starting from this test which has a number of reasonably good properties, and then using the union-intersection principle [1, 2], a test is obtained for compound symmetry, i.e., for H_0: $\sigma_{11} = \sigma_{22} = \ldots = \sigma_{pp}$ and all σ_{ij}'s are equal $(i \neq j = 1, 2, \ldots, p)$, where σ_{ij} is any element of the covariance matrix Σ of a p-variate normal population.

2. **Test Construction**. Let $\rho(x_1, x_2)$ denote the population correlation coefficient between x_1 and x_2 and $r(x_1, x_2)$ the same in a random sample of size, say n, from that population. Then, for a set of stochastic variables $\underline{x}'(1 \times p)(= x_1, \ldots, x_p)$ having a p-variate normal distribution, notice that for any arbitrary non-null vector $\underline{a}'(1 \times p)$, $\underline{a}'(1 \times p) \underline{x}(p \times 1)$ and $\sum_{i=1}^{p} x_i$ have a bivariate normal distribution and, now letting $\sum_{i=1}^{p} a_i = 0$,

consider the hypothesis: $\rho(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) = 0 = H_{0\underline{a}}$ (say). Next notice that

(2.1) H_0: all σ_{ii}'s are equal and all σ_{ij}'s are equal ($i \neq j = 1, \ldots, p$)

$$= \bigcap_{\underline{a}} H_{0\underline{a}} = \bigcap_{\underline{a}} \Big[\rho(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) = 0 \Big] ,$$

where $\underline{a}'(1 \times p)$ is any arbitrary row vector subject to $\sum_{i=1}^{p} a_i = 0$.

Now going back to $H_{0\underline{a}}$, we have for this hypothesis the Pitman critical region, say $W_{\underline{a}}(\alpha)$, of size α, given by

(2.2) $\qquad W_{\underline{a}}(\alpha): \quad r^2(\sum_{i=1}^{p} x_i, \underline{a}' \underline{x}) \geq r_c^2 (n-2),$

where $r_c(n-2)$ is the upper $\alpha/2$-point of the central r-distribution in random samples of size n.

Hence, by the union-intersection heuristic principle $[1, 2]$ we have, for $H_0(= \bigcap_{\underline{a}} H_{0\underline{a}})$, the critical region $W(\beta)$ of size β given by

(2.3) $\qquad W(\beta): \bigcap_{\underline{a}} \Big[r^2(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) \geq r_c^2(n-2) \Big] ,$

i.e., $\text{Sup}_{\underline{a}} \ r^2(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) \geq r_c^2(n-2)$

i.e., $\text{Sup}_{\underline{a}} \ r^2(\sum_{i=1}^{p} x_i, \sum_{i=1}^{p-1} a_i(x_i - x_p)) \geq r_c^2(n-2) ,$

remembering that $\sum_{i=1}^{p} a_i = 0$, i.e., $a_p = -\sum_{i=1}^{p-1} a_i$.

It is easy to check that

$$(2.4) \qquad \operatorname{Sup}_{\underline{a}} \; r^2\left(\sum_{i=1}^{p} x_i, \; \sum_{i=1}^{p-1} a_i (x_i - x_p) \right)$$

= square of the sample multiple correlation between $\sum_{i=1}^{p} x_i$ and the $(p - 1)$-

set of variables $(x_1 - x_p)$, $(x_2 - x_p)$, ..., $(x_{p-1} - x_p)$ =

$R^2 \left[\sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), (x_2 - x_p), \ldots, (x_{p-1} - x_p) \right]$: R^2 (say).

Notice that, since the new p-set also has the p-variate normal distribution, this R has the well-known multiple correlation distribution with degrees of freedom $p - 1$ and $n - p$ and a non-centrality parameter which is the popula-

tion multiple correlation between $\sum_{i=1}^{p} x_i$ and the $(p - 1)$-set above and

which let us call ρ^2. It is easy to check that $\rho = 0$, i.e., that R has the central multiple correlation distribution, if and only if

$\rho\left(\sum_{i=1}^{p} x_i, x_i - x_p \right) = 0$ ($i = 1, 2, \ldots, p - 1$), i.e., if and only if

$\rho\left(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x} \right) = 0$ (for all non-null $\underline{a}$ subject to $\sum_{i=1}^{p} a_i = 0$).

We have thus, for testing compound symmetry, the critical region $W(\beta)$ of size β given by

$(2.5) \qquad R \left[\sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), \ldots, (x_{p-1} - x_p) \right] \geq R_\beta (p - 1, n - p),$

when R_β is the upper β-point of the well-known central multiple correlation distribution.

It can be checked after some little algebra that in terms, respectively, of the elements of the sample and population covariance matrices S and Σ, R and ρ will be given by

$$(2.6) \qquad R^2 = 1 - p \sum_{i=1}^{p} z_i z^i / (\sum_{i=1}^{p} z_i)(\sum_{i=1}^{p} z^i),$$

$$\rho^2 = 1 - p \sum_{i=1}^{p} \zeta_i \zeta^i / (\sum_{i=1}^{p} \zeta_i)(\sum_{i=1}^{p} \zeta^i),$$

when

$$z_i = \sum_{j=1}^{p} s_{ij}, \quad z^i = \sum_{j=1}^{p} s^{ij},$$

$$\zeta_i = \sum_{j=1}^{p} \sigma_{ij} \quad \text{and} \quad \zeta^i = \sum_{j=1}^{p} \sigma^{ij}.$$

Note that $s_{ij} = s_{ji}$, $s^{ij} = s^{ji}$, $\sigma_{ij} = \sigma_{ji}$ and $\sigma^{ij} = \sigma^{ji}$.

The power properties of this test will be discussed in a later note.

References

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